CENTERING PROBLEMS FOR PROBABILITY MEASURES ON FINITE DIMENSIONAL VECTOR SPACES

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Abstract. The paper deals with various centering problems for probability measures on finite dimensional vector spaces. We show that for every such measure there exists a vector h satisfying $\mu * \delta(h) = S(\mu * \delta(h))$ for each symmetry S of μ, generalizing thus Jurek's result obtained for full measures. An explicit form of the h is given for infinitely divisible μ. The main result of the paper consists in the analysis of quasi-decomposable (operator-semistable and operator-stable) measures and finding conditions for the existence of a 'universal centering' of such a measure to a strictly quasi-decomposable one.

Introduction

The general setup for the problems considered in this paper may be formulated as follows. For a finite dimensional vector space V, we are given a class Φ of transformations defined on a subset $\mathcal{S}$ of all probability measures $\mathcal{P}(V)$ on V and taking values in $\mathcal{P}(V)$. Let μ be a measure belonging to $\mathcal{S}$, and denote by $\Phi_0(\mu)$ a subset of Φ consisting of the elements φ having the property

$$\mu = \varphi(\mu) * \delta(h_\varphi),$$

with some $h_\varphi \in V$. We are looking for a 'universal centering' of μ with respect to $\Phi_0(\mu)$, by which is meant an element $h' \in V$, independent of $\varphi \in \Phi_0(\mu)$, such that for all $\varphi \in \Phi_0(\mu)$ we have

$$\mu * \delta(h') = \varphi(\mu * \delta(h').$$

Two cases are dealt with in detail:

1. $\mathcal{S} = \mathcal{P}(V)$, $\Phi = \operatorname{End} V$ and $\varphi(\mu) = \mu \circ \varphi^{-1}$.
2. $\mathcal{S}$ — infinitely divisible measures, $\Phi = \{\varphi = (a, A) : a \in (0, \infty),$ $A \in \operatorname{End} V\}$, and $\varphi(\mu) = (A\mu)^{1/a}$.

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The first case was considered by Z. Jurek in [6] for S being the set of full measures, so that $\Phi_0(\mu)$ is the so-called symmetry group of μ; as for the second, note that in order that $\Phi_0(\mu)$ be nontrivial μ must be (a, A)-quasi-decomposable with some $a \neq 1$ and $A \in \operatorname{End} V$, i.e.,

$$\mu^a = A\mu * \delta(h), \tag{$*$}$$

and our problem consists in centering μ to a strictly quasi-decomposable measure, that is we look for an $h' \in V$ such that

$$(\mu * \delta(h'))^a = A(\mu * \delta(h'))$$

for all pairs (a, A) satisfying $(*)$. In the case when $(*)$ is satisfied by pairs (t, t^B) for all $t > 0$, i.e., when μ is operator-stable, a partial question concerning only existence and not universality has been solved in [13]. However, also in this case, our solution of the general problem is given in a form which appears to be well suited to both (operator-stable as well as operator-semistable) possible situations and is considerably different in form and method from that of [13].

The paper bears a direct connection to the theory of operator-limit distributions on finite dimensional vector spaces. A useful source of information about this theory is monograph [7] to which the reader is referred for additional facts, explanations, comments etc.

1. Preliminaries and notation

Throughout the paper, V will stands for a finite dimensional real vector space with an inner product $(\cdot, \cdot)$ yielding a norm $\|\cdot\|$, and the σ-algebra $\mathcal{B}(V)$ of its Borel subsets. We let $\operatorname{End} V$ denote the set of all linear operators on V, whereas $\operatorname{Aut} V$ stands for the linear invertible operators.

Let $A \colon V \to W$ be a linear mapping into a finite dimensional real vector space W, and let μ be a (probability) measure over $(V, \mathcal{B}(V))$. The measure $A\mu$ on $(W, \mathcal{B}(W))$ is defined by

$$A\mu(E) = \mu(A^{-1}(E)), \qquad E \in \mathcal{B}(W).$$

In particular, if $\xi \colon \Omega \to V$ is a random variable taking values in V and μ is the law of ξ, then $A\mu$ is the law of $A\xi$.

The following equalities are easily verified

$$A(B\mu) = (AB)\mu, \quad \widehat{A\mu}(v) = \hat{\mu}(A^* v), \quad A(\mu * \nu) = A\mu * A\nu,$$

for linear operators A, B and probability measures μ, ν (here $\hat{}$ denotes the characteristic function, and the asterisk $*$ stands for the convolution of measures or for the adjoint of an operator, as the case may be). By $\delta(h)$ we denote the probability measure concentrated at point h.

A probability measure on V is called *full* if it is not concentrated on any proper hyperplane of V. Let μ be a probability measure on V. Then there exists a smallest hyperplane U of V such that μ is concentrated on U and, by a little abuse of language, we can speak of μ being full on U. In this case, there is the unique subspace W of V and an element $h \in V$ such that $U = W + h$. We call W the *supporting subspace* of μ and denote it by $W = \operatorname{ssupp}(\mu)$. It is clear that $\operatorname{ssupp}(\mu) = \{0\}$ if and only if $\mu = \delta(h)$ for some $h \in V$, and $\operatorname{ssupp}(\mu) = V$ if and only if μ is full.

A linear operator S on V is called a *symmetry* of μ if there is an $h \in V$ such that $\mu = S\mu * \delta(h)$. The set of all symmetries of μ is denoted by $\mathbb{A}(\mu)$. Let us recall that if μ is full, then $\mathbb{A}(\mu)$ is a compact subgroup of $\operatorname{Aut} V$ (cf. [7, Corollary 2.3.2] or [14, 15]). We recall that an infinitely divisible measure μ on V has the unique representation as a triple $[m, D, M]$, where $m \in V$, D is a non-negative linear operator on V, and M is the Lévy spectral measure of μ, i.e. a Borel measure defined on $V_0 = V - \{0\}$ such that $\int_{V_0} \|v\|^2/(1 + \|v\|^2)\, M(dv) < \infty$. The characteristic function of μ has then the form

$$\hat{\mu}(u) = \exp\left\{ i(m,u) - \frac{1}{2}(Du,u) + \int_{V_0} \left(e^{i(v,u)} - 1 - \frac{i(v,u)}{1+\|v\|^2} \right) M(dv) \right\}$$

(cf. e.g. [12]). A straightforward calculation shows that for $\mu = [m, D, M]$ and $A \in \operatorname{End} V$, we have $A\mu = [m', ADA^*, AM]$, where

$$m' = Am + \int_{V_0} \frac{\|u\|^2 - \|Au\|^2}{(1+\|Au\|^2)(1+\|u\|^2)} Au\, M(du). \tag{1}$$

One of the main objects considered in this paper is the class of operator-semistable (operator-stable) or, more generally, quasi-decomposable measures. A measure μ on V is called (a, A)-*quasi-decomposable* with $a > 0$, $a \neq 1$, $A \in \operatorname{End} V$, if it is infinitely divisible and

$$\mu^a = A\mu * \delta(h_{a,A}) \qquad \text{for some} \quad h_{a,A} \in V. \tag{2}$$

If $h_{a,A} = 0$, then μ is called *strictly* (a, A)-*quasi-decomposable.* μ is called *quasi-decomposable* if it is (a, A)-quasi-decomposable for some pair (a, A). It is known that a quasi-decomposable measure is operator semistable, i.e. arises as the limit law of a sequence

$$A_n \nu^{k_n} * \delta(h_n),$$

where $\nu \in \mathcal{P}(V)$, $A_n \in \operatorname{End} V$, $h_n \in V$, $k_{n+1}/k_n \to r \geqslant 1$ and the power ν^{k_n} is taken in the sense of convolution, and the converse is true if the limit measure is full (cf. [3, 4, 5, 8, 9, 14] for a more detailed description of this class).

2. Universal centering with respect to symmetries

In this section we show that for any probability measure μ on V there exists an $h' \in V$ such that, for each $S \in \mathbb{A}(\mu)$,

$$\mu * \delta(h') = S(\mu * \delta(h')).$$

As noted in the Introduction, this problem was solved in [6] under the fullness assumption on μ.

In addition to the general solution, we give an explicit form of the h' for μ being infinitely divisible.

Our first lemma is a slight refinement of Proposition 1 from [14].

Lemma 1. *Let W be a subspace of V and denote by $W^\perp$ its orthogonal subspace. Then $|\hat{\mu}(v)| = 1$ for $v \in W^\perp$ if and only if $\mu = \nu * \delta(h)$ with $\nu(W) = 1$ and $h \in W^\perp$.*

Proof. Assume that $|\hat{\mu}(v)| = 1$ for $v \in W^\perp$ and let P be the orthogonal projection on $W^\perp$. Then $P\mu$ is a measure concentrated on $W^\perp$ and for any $v \in V$, we have

$$|\widehat{P\mu}(v)| = |\hat{\mu}(Pv)| = 1,$$

thus $P\mu = \delta(h)$ for some $h \in W^\perp$, which gives the equality

$$\mu = P\mu * \mu * \delta(-h).$$

On account of [15, Proposition 1.5] or [7, Theorem 2.3.6(b)] we have

$$\mu = P\mu * (I - P)\mu,$$

where I is the identity operator, and putting $\nu = (I - P)\mu$, we get the formula $\mu = \nu * \delta(h)$ with ν concentrated on W.

Conversely, if $\mu = \nu * \delta(h)$ with ν concentrated on W, then $\hat{\nu}(v) = 1$ for $v \in W^\perp$, and so $|\hat{\mu}(v)| = |\hat{\nu}(v)| = 1$ for $v \in W^\perp$. □

Lemma 2. *Let W be a subspace of V. Assume that μ is concentrated on W and the decomposition $\mu = \nu * \lambda$ holds. Then $\nu = \nu_1 * \delta(h)$, $\lambda = \lambda_1 * \delta(-h)$, where ν_1 and λ_1 are concentrated on W, and $h \in W^\perp$.*

Proof. For $v \in W^\perp$ we have

$$1 = |\hat{\mu}(v)| = |\hat{\nu}(v)||\hat{\lambda}(v)|,$$

and thus

$$|\hat{\nu}(v)| = |\hat{\lambda}(v)| = 1.$$

By Lemma 1, $\nu = \nu_1 * \delta(h)$, $\lambda = \lambda_1 * \delta(h')$ for some $h, h' \in W^\perp$, where ν_1 and λ_1 are concentrated on W. Moreover,

$$\mu = \mu * \lambda = \nu_1 * \lambda_1 * \delta(h + h'),$$

which yields $h + h' \in W$, so $h + h' = 0$ and the assertion follows. □

The next proposition gives an important property of the supporting subspaces in the case of a decomposition of measures.

Proposition 3. *Let $A \in \operatorname{End} V$, and let μ, ν, λ be probability measures on V such that*

$$\mu = A\nu * \lambda.$$

Put $W = \operatorname{ssupp}(\mu)$, $U = \operatorname{ssupp}(\nu)$. Then $A(U) \subset W$.

Proof. There is an $h \in V$ such that ν is full on $U + h$. We have

$$\mu * \delta(-h) = A(\nu * \delta(-h)) * \lambda * \delta(Ah - h),$$

and putting

$$\mu' = \mu * \delta(-h), \quad \nu' = \nu * \delta(-h), \quad \lambda' = \lambda * \delta(Ah - h)$$

we get

$$\mu' = A\nu' * \lambda',$$

moreover, the measure ν' is full on U. We claim that $A\nu'$ is full on $A(U)$. Indeed, let X be a subspace of $A(U)$ and let $x_0 = Au_0$ be an element in $A(U)$ such that $A\nu'$ is concentrated on $X + x_0$. Then

$$1 = A\nu'(X + x_0) = \nu'(A^{-1}(X + x_0)) = \nu'(A^{-1}(X) + u_0),$$

and the fullness of ν' on U yields $U \subset A^{-1}(X)$, thus

$$X \subset A(U) \subset AA^{-1}(X) = X,$$

showing that $X = A(U)$ and, consequently, $A\nu'$ is full on $A(U)$. On account of Lemma 2, there is a $v_0 \in W^{\perp}$ such that

$$(A\nu' * \delta(v_0))(W) = 1,$$

and thus

$$A\nu'(W - v_0) = 1,$$

consequently,

$$A\nu'(A(U) \cap (W - v_0)) = 1.$$

But $A(U) \cap (W - v_0)$ is a hyperplane in $A(U)$, so the fullness of $A\nu'$ on $A(U)$ implies that

$$A(U) \cap (W - v_0) = A(U).$$

Hence $A(U) \subset W - v_0$, which shows that $v_0 \in W$, and finally, $A(U) \subset W$, finishing the proof. $\square$

As an easy consequence of the above proposition and Jurek's result we get the following theorem on the existence of universal centering with respect to $\mathbb{A}(\mu)$ for any probability measure μ on V.

Theorem 4. *Let μ be a probability measure on V. Then there exists $h' \in V$ such that for each $S \in \mathbb{A}(\mu)$*

$$\mu * \delta(h') = S(\mu * \delta(h')).$$

Proof. Let $W = \operatorname{ssupp}(\mu)$. Choose h_0 such that the measure $\mu' = \mu * \delta(h_0)$ is concentrated (and full) on W. We have

$$\mathbb{A}(\mu') = \{S \in \operatorname{End} V : \mu' = S\mu' * \delta(h) \text{ for some } h \in V\},$$

and Proposition 3 yields that μ' is concentrated on W which implies that the h occurring in the definition of $\mathbb{A}(\mu')$ must be in W. Consider μ' only on the subspace W. Then, again by virtue of Proposition 3, we have

$$\mathbb{A}(\mu'|_W) = \{S|_W : S \in \mathbb{A}(\mu')\}.$$

Since μ' is full on W, we infer, on account of [6], that there exists $h'' \in W$ such that for each $S \in \mathbb{A}(\mu')$

$$\mu' * \delta(h'') = (S|_W)(\mu' * \delta(h'')).$$

Now

$$\mu' * \delta(h'') = \mu * \delta(h_0) * \delta(h'') = \mu * \delta(h_0 + h'')$$

and

$$\begin{aligned}(S|_W)(\mu' * \delta(h'')) &= S\mu' * \delta(Sh'') = S(\mu * \delta(h_0)) * \delta(Sh'') \\ &= S(\mu * \delta(h_0 + h'')).\end{aligned}$$

Putting

$$h' = h_0 + h'',$$

we obtain thus

$$\mu * \delta(h') = S(\mu * \delta(h'))$$

for each $S \in \mathbb{A}(\mu')$, and since clearly $\mathbb{A}(\mu') = \mathbb{A}(\mu)$, the conclusion follows. □

Now we shall find the form of a universal centering for any infinitely divisible measure. Let $\mu = [m, D, M]$ be such a measure, and assume first that μ is full, $m = 0$, and $\mathbb{A}(\mu)$ is a subgroup of the orthogonal group $\mathbb{O}$ in V. For each $S \in \mathbb{A}(\mu)$ we then have $S\mu = [m', SDS^*, SM] = [m', D, M]$ where, by virtue of (1),

$$m' = \int_{V_0} \frac{\|u\|^2 - \|Su\|^2}{(1 + \|Su\|^2)(1 + \|u\|^2)} Su\, M(du) = 0,$$

since S is an isometry. Thus $S\mu = \mu$, i.e. any measure $\mu = [0, D, M]$ having the property $\mathbb{A}(\mu) \subset \mathbb{O}$ is itself universally centered. Now let us

assume only that $\mu = [m, D, M]$ is full. Since $\mathbb{A}(\mu)$ is compact, there exists an invertible operator T on V such that

$$T\mathbb{A}(\mu)T^{-1} \subset \mathbb{O}. \tag{3}$$

It is easily seen that

$$T\mathbb{A}(\mu)T^{-1} = \mathbb{A}(T\mu). \tag{4}$$

The measure $T\mu$ has the form $T\mu = [m', TDT^*, TM]$ with

$$m' = Tm + \int_{V_0} \frac{\|u\|^2 - \|Tu\|^2}{(1+\|Tu\|^2)(1+\|u\|^2)} Tu\, M(du). \tag{5}$$

According to the first part of our considerations, $-m'$ is a universal centering for $T\mu$. Let $S \in \mathbb{A}(\mu)$. Then $TST^{-1} \in \mathbb{A}(T\mu)$, and we have

$$T\mu * \delta(-m') = TST^{-1}(T\mu * \delta(-m')) = TS(\mu * \delta(-T^{-1}m')),$$

which yields the equality

$$\mu * \delta(-T^{-1}m') = S(\mu * \delta(-T^{-1}m')),$$

meaning that $h' = T^{-1}m'$ is a universal centering for μ. From (5) we get the formula

$$h' = -\left(m + \int_{V_0} \frac{\|u\|^2 - \|Tu\|^2}{(1+\|Tu\|^2)(1+\|u\|^2)}\, M(du)\right).$$

Finally, let $\mu = [m, D, M]$ be an arbitrary infinitely divisible measure on V. Put $W = \mathrm{ssupp}(\mu)$, and let $h_0 \in V$ be such that $\mu' = \mu * \delta(h_0) = [m + h_0, D, M]$ is full on W. The preceding discussion applied to the measure μ' on W shows that a universal centering h'' for μ' has the form

$$h'' = -\left(m + h_0 + \int_{W_0} \frac{\|u\|^2 - \|Tu\|^2}{(1+\|Tu\|^2)(1+\|u\|^2)} u\, M(du)\right),$$

where T is an invertible operator on W such that $T\mathbb{A}(\mu')T^{-1}$ is a subgroup of the isometries on W. It is clear that $h' = h'' + h_0$ is a universal centering for μ, so for this centering we have the formula

$$h' = -\left(m + \int_{W_0} \frac{\|u\|^2 - \|Tu\|^2}{(1+\|Tu\|^2)(1+\|u\|^2)} u\, M(du)\right),$$

where $W = \mathrm{ssupp}(\mu)$, and T is an invertible operator on W such that $T(\mathbb{A}(\mu)|\, W)T^{-1}$ is a subgroup of the isometries on W.

3. Centering problem for quasi-decomposable measures

For an infinitely divisible measure μ on V and $a > 0$ put, following [14] (cf. also [7, p. 187]),

$$G_a(\mu) = \{A \in \operatorname{End} V : \mu^a = A\mu * \delta(h) \quad \text{for some } h \in V\}.$$

We recall that μ is quasi-decomposable if $G_a(\mu) \neq \emptyset$ for some $a \neq 1$. In this section, given a quasi-decomposable measure μ, we aim at finding conditions for the existence of an $\hat{h} \in V$ such that for any $a > 0$ with $G_a(\mu) \neq \emptyset$ and any $A \in G_a(\mu)$ the following equality holds

$$(\mu * \delta(\hat{h}))^a = A(\mu * \delta(\hat{h})). \tag{6}$$

μ is then said to have a *universal quasi-decomposability centering*.

So, let us assume that $G_a(\mu) \neq \emptyset$ for some $a \neq 1$, i.e., that for μ equality (2) holds. Thus if we have (6) with some $\hat{h} \in V$, then

$$\mu^a * \delta(a\hat{h}) = A\mu * \delta(A\hat{h}),$$

yielding, by (2), the equality

$$A\mu * \delta(h_{a,A} + a\hat{h}) = A\mu * \delta(A\hat{h}),$$

which means that

$$h_{a,A} = A\hat{h} - a\hat{h}. \tag{7}$$

On the other hand, it is immediately seen that (7) implies (6) under the assumption of the (a, A)-quasi-decomposability of μ, so for such μ we have equivalence of (6) and (7). Thus our task consists in finding conditions for the existence of a solution $\hat{h}$ of equation (7) and showing that this solution is independent of a and A.

First we address the problem of the universality of centering. This will be performed in two steps. In the first one we shall show that if, for a given a for which formula (2) holds, there is an $\hat{h}_0$ satisfying (6) (or (7)) for some $A_0 \in G_a(\mu)$, then there is an $\hat{h}$ satisfying (6) for all $A \in G_a(\mu)$. In the second step, we prove that the existence of centering for some a yields the existence of centering for all the a that can occur in formula (2), thus that this centering is universal.

In the first part of our considerations we may assume, in view of Proposition 3 and the obvious fact that the existence of universal quasi-decomposability centering is not affected by shifts, that μ is full. We then have

Lemma 5. *Assume that μ is full. Then for any $A \in G_a(\mu)$ the mappings $G_a(\mu) \ni B \mapsto B^{-1}A$ and $G_a(\mu) \ni B \mapsto AB^{-1}$ are bijections from $G_a(\mu)$ onto $\mathbb{A}(\mu)$.*

Proof. It is easily seen that $B^{-1} \in G_{1/a}(\mu)$ for $B \in G_a(\mu)$, and thus

$$\mu^{1/a} = B^{-1}\mu * \delta(h_{1/a,B^{-1}}),$$

giving the equalities

$$\begin{aligned}\mu &= B^{-1}\mu^a * \delta(ah_{1/a,B^{-1}}) = B^{-1}(A\mu * \delta(h_{a,A})) * \delta(ah_{1/a,B^{-1}}) \\ &= B^{-1}A\mu * \delta(B^{-1}h_{a,A} + ah_{1/a,B^{-1}}),\end{aligned}$$

which shows that $B^{-1}A \in \mathbb{A}(\mu)$. For any $S \in \mathbb{A}(\mu)$, $A \in G_a(\mu)$, we have $S^{-1} \in \mathbb{A}(\mu)$, so the operator $B = AS^{-1}$ belongs to $G_a(\mu)$ and

$$S = B^{-1}A,$$

showing that the mapping $B \mapsto B^{-1}A$ is onto $\mathbb{A}(\mu)$. Since it is injective the conclusion follows. Analogously we deal with the case of the mapping $B \mapsto AB^{-1}$. □

The above mentioned fact that the existence of universal quasi-decomposability centering is not affected by shifts allows us to assume further that μ is universally centered with respect to $\mathbb{A}(\mu)$. This assumption is made in the remainder of the paper.

Lemma 6. *For any $A, B \in G_a(\mu)$ we have $A\mu = B\mu$ and $h_{a,A} = h_{a,B}$.*

Proof. The following equality holds

$$A\mu * \delta(h_{a,A}) = B\mu * \delta(h_{a,B}),$$

which gives

$$B^{-1}A\mu * \delta(B^{-1}h_{a,A}) = \mu * \delta(B^{-1}h_{a,B}),$$

that is

$$\mu = B^{-1}A\mu * \delta(B^{-1}(h_{a,A} - h_{a,B})).$$

Since $B^{-1}A \in \mathbb{A}(\mu)$ and μ is universally centered with respect to $\mathbb{A}(\mu)$, we get

$$B^{-1}(h_{a,A} - h_{a,B}) = 0,$$

consequently, $h_{a,A} = h_{a,B}$ and $A\mu = B\mu$. □

The lemma above says that, with μ universally centered with respect to $\mathbb{A}(\mu)$, we have the equality

$$\mu^a = A\mu * \delta(h_a), \qquad A \in G_a(\mu),$$

with the same h_a for all $A \in G_a(\mu)$. This yields an important property of the h_a.

Lemma 7. *For each $S \in \mathbb{A}(\mu)$, we have $Sh_a = h_a$.*

Proof. We have, for $S \in \mathbb{A}(\mu)$,

$$\mu^a = S\mu^a = SA\mu * \delta(Sh_a),$$

moreover, since by Lemma 5, $SA \in G_a(\mu)$, it follows that

$$\mu^a = SA\mu * \delta(h_a),$$

which proves the claim. □

Finally, let us make our last simplification. For $T \in \operatorname{Aut} V$ we clearly have

$$G_a(T\mu) = TG_a(\mu)T^{-1},$$

so μ is (a, A)-quasi-decomposable if and only if $T\mu$ is (a, TAT^{-1})-quasi-decomposable; moreover, equality (6) is equivalent to the equality

$$(T\mu * \delta(T\hat{h}))^a = TAT^{-1}(T\mu * \delta(T\hat{h})).$$

Therefore $\hat{h}$ is a universal quasi-decomposability centering of μ if and only if $T\hat{h}$ is a universal quasi-decomposability centering of $T\mu$. Now taking T such that (3) and (4) hold, the above considerations allow us to assume that $\mathbb{A}(\mu) \subset \mathbb{O}$. Let

$$W = \{v : Sv = v, \quad S \in \mathbb{A}(\mu)\}$$

be the fixed-point space for $\mathbb{A}(\mu)$, and let P be the orthogonal projection onto W.

Proposition 8. *For each $A \in G_a(\mu)$, we have $AP = PA$.*

Proof. Take arbitrary $A, B \in G_a(\mu)$. Since $B^{-1}A \in \mathbb{A}(\mu)$, we get for each $v \in W$

$$B^{-1}Av = v,$$

giving

$$Av = Bv. \tag{8}$$

For any $S \in \mathbb{A}(\mu)$ we have $SA \in G_a(\mu)$, thus if $v \in W$, then

$$SAv = Av,$$

that is

$$A(W) \subset W,$$

or, equivalently,

$$PAP = AP. \tag{9}$$

Now put $S = AB^{-1}$. Then $S \in \mathbb{A}(\mu)$, and since $\mathbb{A}(\mu)$ is a subgroup of the orthogonal group, we get

$$S^{-1} = S^* = B^{*-1}A^* \in \mathbb{A}(\mu),$$

which, as in the first part of the proof, yields

$$A^*v = B^*v, \quad v \in W.$$

For any $S \in \mathbb{A}(\mu)$, we have

$$(SA^*)^* = AS^* = AS^{-1} \in G_a(\mu),$$

and hence

$$SA^*v = A^*v \quad v \in W,$$

giving the equality

$$PA^*P = A^*P.$$

Upon taking adjoints, we obtain

$$PAP = PA,$$

which, together with (9), gives the desired result. □

Now we are in a position to prove the universality of centering with respect to $G_a(\mu)$, under the assumption of the existence of a centering for an operator from $G_a(\mu)$.

Proposition 9. *Assume that for some $A_0 \in G_a(\mu)$ there is an $\hat{h}_0$ such that*

$$(\mu * \delta(\hat{h}_0))^a = A_0(\mu * \delta(\hat{h}_0)). \tag{10}$$

Then there exists $\hat{h}$ such that for all $A \in G_a(\mu)$ equality (6) holds. Moreover, $\hat{h}$ is also a universal centering with respect to $\mathbb{A}(\mu)$.

Proof. As we have shown before, equality (10) is equivalent to the equality

$$h_a = A_0\hat{h}_0 - a\hat{h}_0$$

and as $h_a \in W$ by Lemma 7, we get

$$h_a = Ph_a = PA_0\hat{h}_0 - aP\hat{h}_0 = A_0P\hat{h}_0 - aP\hat{h}_0.$$

Putting

$$\hat{h} = P\hat{h}_0,$$

we obtain

$$h_a = A_0\hat{h} - a\hat{h},$$

moreover, since $\hat{h} \in W$, we have by (8)

$$A_0\hat{h} = A\hat{h}$$

for all $A \in G_a(\mu)$, which leads to the equality

$$h_a = A\hat{h} - a\hat{h}, \qquad A \in G_a(\mu),$$

proving the first part of the claim. The second part follows from the first and Lemma 5. □

For our further analysis, it will be convenient to rewrite condition (7) in a slightly different form. Let $T \in \operatorname{End} V$ and let $\mathcal{N}(T)$ denote its null space, i.e.

$$\mathcal{N}(T) = \{v \in V : Tv = 0\}.$$

From elementary Hilbert space theory and the finite dimensionality of V, we have the following orthogonal decomposition

$$V = \mathcal{N}(T^*) \oplus T(V). \tag{11}$$

Now condition (7) means simply that $h_{a,A} \in (A - aI)(V)$, which by (11) is equivalent to

$$h_{a,A} \perp \mathcal{N}(A^* - aI), \tag{12}$$

which is the form we shall employ.

Now we shall analyze the universality with respect to various a's that can occur in formula (2). According to [10, Theorem 3.2] there are two possibilities: either

(i) $a = c^n$ for a unique $0 < c < 1$ and some integer n,

or

(ii) a may be an arbitrary positive real number, in which case for μ the following formula holds

$$\mu^t = t^B \mu * \delta(h_{t,B}), \qquad t > 0 \tag{13}$$

for some $B \in \operatorname{End} V$ and t^B defined as $t^B = e^{(\log t)B}$, that is, μ is operator-stable. We shall call these two cases discrete and continuous, respectively, and shall deal with them separately.

Discrete case. According to our previous considerations we may assume that μ is centered universally with respect to $\mathbb{A}(\mu)$. Then

$$\mu^c = A\mu * \delta(h_c) \quad \text{for each} \quad A \in G_c(\mu),$$

and iterating the equality above, we obtain

$$\mu^{c^n} = A^n \mu * \delta(c^{n-1}h_c + c^{n-2}Ah_c + \cdots + A^{n-1}h_c)$$

and

$$\mu^{(1/c)^n} = A^{-n}\mu * \delta((1/c)^{n-1}h_{1/c} + (1/c)^{n-2}A^{-1}h_{1/c} + \cdots + (A^{-1})^{n-1}h_{1/c})$$

for all positive integers n. Denoting

$$h_n = h_{c^n}, \quad p_n(b, A) = b^{n-1}I + b^{n-2}A + \cdots + A^{n-1}, \quad n = 1, 2, \ldots,$$

we get the formulas

$$h_n = p_n(c, A)h_1, \quad h_{-n} = p_n(c^{-1}, A^{-1})h_{-1}, \qquad n = 1, 2, \ldots;$$

moreover, it is immediately seen that

$$h_1 = -cAh_{-1}. \tag{14}$$

Now we are in a position to set the problem of the universality of centering together with an important point on its existence.

Proposition 10. *There exists a universal quasi-decomposability centering for μ if and only if for some integer n and $A_n \in G_{c^n}(\mu)$ there exists a centering of μ with respect to the pair (c^n, A_n).*

Proof. Assume that n is positive and that μ may be centered with respect to the pair (c^n, A_n) with some $A_n \in G_{c^n}(\mu)$. On account of Proposition 9 we may assume that this centering is universal with respect to the whole of $G_{c^n}(\mu)$. Take an arbitrary $A \in G_c(\mu)$. Then $A^n \in G_{c^n}(\mu)$ and since

$$\mu^{c^n} = A^n \mu * \delta(h_n),$$

the existence of a centering for (c^n, A_n) yields the condition

$$h_n \perp \mathcal{N}(A^{*n} - c^n I).$$

For each $v \in \mathcal{N}(A^* - cI)$ we have

$$p_n(c, A)^* v = p_n(c, A^*) v = c^{n-1} v + c^{n-2} A^* v + \cdots + A^{*n-1} v = nc^{n-1} v,$$

and since

$$\mathcal{N}(A^* - cI) \subset \mathcal{N}(A^{*n} - c^n I),$$

we get

$$\begin{aligned} 0 &= (h_n, v) = (p_n(c, A) h_1, v) = (h_1, p_n(cA)^* v) \\ &= nc^{n-1}(h_1, v), \qquad v \in \mathcal{N}(A^* - cI), \end{aligned}$$

which means that

$$h_1 \perp \mathcal{N}(A^* - cI). \tag{15}$$

For n negative, we would obtain, considering the pair (c^{-1}, A^{-1}) instead of (c, A), the condition

$$h_{-1} \perp \mathcal{N}(A^{-1*} - c^{-1} I),$$

which, by (14), gives again condition (15). But this condition together with Proposition 9 say that there is a centering $\hat{h}$ universal with respect to $G_a(\mu)$. Thus

$$(\mu * \delta(\hat{h}))^c = A(\mu * \delta(\hat{h})) \qquad \text{for each} \quad A \in G_a(\mu),$$

and, consequently,

$$(\mu * \delta(\hat{h}))^{c^n} = A^n(\mu * \delta(\hat{h})) \quad n = 0, \pm 1, \ldots.$$

For any $A_n \in G_{c^n}(\mu)$, we have $A_n = A^n S$ with some $S \in \mathbb{A}(\mu)$, and since $\hat{h}$ is also universal with respect to $\mathbb{A}(\mu)$, we have

$$A_n(\mu * \delta(\hat{h})) = A^n S(\mu * \delta(\hat{h})) = A^n(\mu * \delta(\hat{h})) = (\mu * \delta(\hat{h}))^{c^n},$$

showing the universality of centering. □

Continuous case. First, notice that formulas (1) and (2) lead to the following equality for the shift for $\mu = [m, D, M]$

$$h_{a,A} = am - Am - \int_{V_0} \frac{\|u\|^2 - \|Au\|^2}{(1 + \|Au\|^2)(1 + \|u\|^2)} Au\, M(du), \tag{16}$$

which for μ satisfying (13) takes the form

$$h_{t,B} = tm - t^B m - \int_{V_0} \frac{\|u\|^2 - \|t^B u\|^2}{(1 + \|t^B u\|^2)(1 + \|u\|^2)} t^B u\, M(du).$$

Put, for the sake of convenience,

$$f_B(t) = h_{e^t,B}, \qquad t \in \mathbb{R}.$$

Then we have for f_B

$$f_B(t) = e^t m - e^{tB} m + \int_{V_0} \frac{\|e^{tB}u\|^2 - \|u\|^2}{(1 + \|u\|^2)(1 + \|e^{tB}u\|^2)} e^{tB} u\, M(du). \tag{17}$$

For each fixed $u \in V$, consider the function

$$g(t) = \|e^{tB}u\|^2, \qquad t \in \mathbb{R}.$$

We have

$$g'(t) = 2(Be^{tB}u, e^{tB}u), \qquad t \in \mathbb{R}, \tag{18}$$

and since $e^{tB}u \to u$ as $t \to 0$, we get for sufficiently small t's

$$\left| \frac{\|e^{tB}u\|^2 - \|u\|^2}{t} \right| \leqslant C\|B\|\|u\|^2$$

which gives the following estimation

$$\left\| \frac{1}{t} \frac{\|e^{tB}u\|^2 - \|u\|^2}{(1 + \|u\|^2)(1 + \|e^{tB}u\|^2)} u \right\| \leqslant \frac{C\|B\|\|u\|^3}{(1 + \|u\|^2)(1 + \frac{1}{2}\|u\|^2)}.$$

But the function on the right-hand side is M-integrable, thus by Lebesgue's theorem we may pass to the limit with $t \to 0$ under the integral sign in the following expression

$$\begin{aligned} \frac{f_B(t)}{t} &= -\frac{e^{tB} - e^t I}{t} m \\ &\quad + \int_{V_0} \frac{\|e^{tB}u\|^2 - \|u\|^2}{t} \frac{e^{tB}u}{(1 + \|e^{tB}u\|^2)(1 + \|u\|^2)} M(du) \end{aligned}$$

and obtain, taking into account (18),

$$v_0 = \lim_{t\to 0} \frac{f_B(t)}{t} = (I-B)m + \int_{V_0} \frac{2(Bu,u)}{(1+\|u\|^2)^2} u\, M(du). \tag{19}$$

Since $f_B(0) = 0$, we have

$$v_0 = f_B'(0).$$

Now, according to [14, Formula (8.2), p. 64] or [7, Sec.4.9, p. 236], the function h satisfies the following equation

$$h_{st,B} = t^B h_{s,B} + s h_{t,B}, \qquad s,t > 0, \tag{20}$$

which implies that for f_B we have

$$f_B(s+t) = e^{tB} f_B(s) + e^s f_B(t), \qquad s,t \in \mathbb{R}. \tag{21}$$

In [14] (cf. also [7]) equation (20) is solved in general, under the assumption $1 \notin \operatorname{sp} B$. We shall find the form of the function f_B without any restrictions on the spectrum (however, it should be kept in mind that we do have the existence of $f_B'(0)$ at our disposal).

Lemma 11. *The function f_B has the form*

$$f_B(t) = e^t \int_0^t e^{s(B-I)} v_0\, ds, \tag{22}$$

where $v_0 = f_B'(0)$ is given by equality (19).

Proof. For each fixed t and any s we have

$$\frac{f_B(t+s) - f_B(t)}{s} = e^{tB} \frac{f_B(s)}{s} + \frac{e^s - 1}{s} f_B(t),$$

and passing to the limit with $s \to 0$ yields the equation

$$f_B'(t) = e^{tB} v_0 + f_B(t). \tag{23}$$

It follows from e.g. [1, Chapter 10, p. 169] that the general solution of (23) has the form

$$f_B(t) = e^t u_0 + \int_0^t e^{t-s} e^{sB} v_0\, ds,$$

and taking into account our initial condition $f_B(0) = 0$ we get (22). □

The next proposition sets the problem of the universality of centering; it also adds an important point in the question of existence.

Proposition 12. *The following conditions are equivalent:*

(i) $v_0 \perp \mathcal{N}(B^* - I)$;
(ii) *there exists a universal centering;*
(iii) *there exists a centering for some $t' > 0$, $t' \neq 1$.*

Proof. (i) $\Longrightarrow$ (ii) By virtue of decomposition (11), we have

$$v_0 = (B-I)v_1 \qquad \text{for some} \quad v_1 \in V,$$

and accordingly

$$f_B(t) = e^t \int_0^t e^{s(B-I)}(B-I)v_1\, ds.$$

But the function $s \mapsto e^{s(B-I)}(B-I)v_1$ under the integral sign is the derivative of the function $s \mapsto e^{s(B-I)}v_1$, so we get

$$f_B(t) = e^t[e^{t(B-I)}v_1 - v_1] = e^{tB}v_1 - e^t v_1,$$

that is

$$h_{t,B} = t^B v_1 - t v_1,$$

which means that v_1 is a universal centering.

(ii) $\Longrightarrow$ (iii) Obvious.

(iii) $\Longrightarrow$ (i) Consider the operator-valued function

$$H(t) = \int_0^t e^{s(B-I)}\, ds, \qquad t \in \mathbb{R}.$$

Then

$$f_B(t) = e^t H(t) v_0,$$

and for any $v \in \mathcal{N}(B^* - I)$, $s \in \mathbb{R}$,

$$e^{s(B^*-I)}v = v.$$

Accordingly, for such v's

$$H(t)^* v = \int_0^t e^{s(B^*-I)} v\, ds = tv,$$

which gives

$$(f_B(t), v) = (e^t H(t) v_0, v) = e^t (v_0, H(t)^* v) = t e^t (v_0, v). \tag{24}$$

Now let a centering for some $t' > 0$, $t \neq 1$ be given. This means that

$$h_{t',B} = t'^B v_1 - t' v_1$$

for some $v_1 \in V$, or with $t_0 = \log t' \neq 0$,

$$f_B(t_0) = e^{t_0 B} v_1 - e^{t_0} v_1 = (e^{t_0 B} - e^{t_0} I) v_1.$$

The last equality is, on account of decomposition (11), equivalent to the relation

$$f_B(t_0) \perp \mathcal{N}(e^{t_0 B^*} - e^{t_0} I),$$

and since

$$\mathcal{N}(B^* - I) \subset \mathcal{N}(e^{t_0 B^*} - e^{t_0} I),$$

we get

$$f_B(t_0) \perp \mathcal{N}(B^* - I).$$

Taking into account (24) we obtain for each $v \in \mathcal{N}(B^* - I)$

$$0 = (f_B(t_0), v) = t_0 e^{t_0}(v_0, v),$$

so $(v_0, v) = 0$ and $v_0 \perp \mathcal{N}(B^* - I)$. □

What we are left with now is the existence problem. Again, as in the analysis of universality, it will be useful to distinguish the discrete and continuous cases, although, as we shall see, there is a remarkable similarity between them.

For a more detailed analysis we shall need a description of the Lévy measure M, which can be found in [9] (discrete case) and in [3, 5, 7] (continuous case). To keep this paper as self-contained as possible, we describe below the main points.

Discrete case. Considering, if necessary, $1/a$ instead of a we may assume that $a < 1$, and further that $\|A\| < 1$. Then, putting

$$Z_A = \{v : \|v\| \leqslant 1 \quad \text{and} \quad \|A^{-1}v\| > 1\},$$

we have the following representation for M

$$M(E) = \sum_{n=-\infty}^{\infty} a^{-n} M(A^{-n}E \cap Z_A), \quad E \text{ — Borel subset of } V_0, \tag{25}$$

i.e., M is determined by its restriction to Z_A which, in turn, may be an (almost) arbitrary finite Borel measure (see Remark 1 below). Formula (25) can be rewritten in the form

$$M(E) = \sum_{n=-\infty}^{\infty} a^{-n} \int_{Z_A} \mathbf{1}_E(A^n u)\, M(du), \tag{26}$$

and for any M-integrable function f on V_0 we have

$$\int_{V_0} f(u)\, M(du) = \sum_{n=-\infty}^{\infty} a^{-n} \int_{Z_A} f(A^n u)\, M(du). \tag{27}$$

Remark 1. The only restriction to the arbitrariness of $M|\, Z_A$ lies in the fact that, in general, the measure M is concentrated on some subspace of V determined by eigenvalues of A. More precisely, if we put $W = \mathrm{ssupp}(\mu)$, then we have a decomposition

$$W = X \oplus Y,$$

where X and Y are A-invariant and such that

$$\operatorname{sp}(A|\,X) \subset \{z \in \mathbb{C} : |z|^2 < a\},$$
$$\operatorname{sp}(A|\,Y) \subset \{z \in \mathbb{C} : |z|^2 = a\};$$

and M must be concentrated on X (see [4, 9] for details). Similar remarks apply to the measure K_B in the continuous case below.

Continuous case. Put

$$L_B = \{v : \|v\| = 1 \quad \text{and} \quad \|t^B v\| > 1 \quad \text{for} \quad t > 1\},$$

and define the mixing measure K_B on the Borel subsets E of L_B by

$$K_B(E) = M(\{t^B v : v \in E, \quad t \geqslant 1\}).$$

Then we have the following continuous counterpart of (26)

$$M(E) = \int_{L_B} \int_0^\infty \mathbf{1}_E(t^B u)\, \frac{dt}{t^2}\, K_B(du), \tag{28}$$

and for any M-integrable function f on V_0

$$\int_{V_0} f(u)\, M(du) = \int_{L_B} \int_0^\infty f(t^B u)\, \frac{dt}{t^2}\, K_B(du).$$

Substituting in the last formula t in place of $\log t$, we get

$$\int_{V_0} f(u)\, M(du) = \int_{L_B} \int_{-\infty}^\infty f(e^{tB} u) e^{-t}\, dt\, K_B(du). \tag{29}$$

Now, we can formulate our final result.

Theorem 13. *Let μ be a quasi-decomposable measure. Put*

$$W_1 = \{v : A^* v = av\} = \mathcal{N}(A^* - aI),$$

if for μ the discrete case holds, and

$$W_2 = \{v : B^* v = v\} = \mathcal{N}(B^* - I),$$

if for μ the continuous case holds. Denote in these two cases

$$N_1 = Z_A, \qquad \nu_1 = M|\, Z_A, \qquad N_2 = L_B, \qquad \nu_2 = K_B.$$

Then there is a universal quasi-decomposability centering for μ if and only if

$$\int_{N_i} (u, w_i)\, \nu_i(du) = 0 \qquad \textit{for all} \quad w_i \in W_i, \tag{30}$$

where $i = 1$ or 2, as the case may be.

Proof. Discrete case. The condition for the existence of a universal centering is given by (12), which by virtue of formula (16) is equivalent to

$$\int_{V_0} \frac{\|u\|^2 - \|Au\|^2}{(1+\|Au\|^2)(1+\|u\|^2)} u\, M(du) \;\perp\; \mathcal{N}(A^* - aI). \tag{31}$$

For each $w \in W_1$ we have by (27)

$$\begin{aligned}
&\Big(\int_{V_0} \frac{\|u\|^2 - \|Au\|^2}{(1+\|Au\|^2)(1+\|u\|^2)} u\, M(du), w\Big)\\
=&\Big(\int_{V_0} \Big(\frac{1}{1+\|Au\|^2} - \frac{1}{1+\|u\|^2}\Big) u\, M(du), w\Big)\\
=&\sum_{n=-\infty}^{\infty} a^{-n}\Big(\int_{Z_A} \Big(\frac{1}{1+\|A^{n+1}u\|^2} - \frac{1}{1+\|A^n u\|^2}\Big) A^n u\, M(du), w\Big)\\
=&\sum_{n=-\infty}^{\infty} a^{-n}\Big(\int_{Z_A} \Big(\frac{1}{1+\|A^{n+1}u\|^2} - \frac{1}{1+\|A^n u\|^2}\Big) u\, M(du), A^{*n} w\Big)\\
=&\Big(\int_{Z_A} \sum_{n=-\infty}^{\infty} \Big(\frac{1}{1+\|A^{n+1}u\|^2} - \frac{1}{1+\|A^n u\|^2}\Big) u\, M(du), w\Big)\\
=&\Big(\int_{Z_A} \Big(\lim_{n\to\infty} \frac{1}{1+\|A^{n+1}u\|^2} - \lim_{n\to\infty} \frac{1}{1+\|A^{-n}u\|^2}\Big) u\, M(du), w\Big)\\
=&\Big(\int_{Z_A} u\, M(du), w\Big)
\end{aligned}$$

since $\|A^n\| \leqslant \|A\|^n \to 0$ and

$$\|A^{-n}u\| \geqslant \frac{\|u\|}{\|A\|^n} \qquad \text{for} \quad n > 0.$$

Thus (31) is equivalent to (30) in the discrete case.

Continuous case. By Proposition 12 and formula (19) the existence of a universal centering is equivalent to

$$\int_{V_0} \frac{2(Bu,u)}{(1+\|u\|^2)^2} u\, M(du) \;\perp\; \mathcal{N}(B^* - I). \tag{32}$$

For each $w \in W_2$ we have by (29)

$$\begin{aligned}
&\Big(\int_{V_0} \frac{2(Bu,u)}{(1+\|u\|^2)^2} u\, M(du), w\Big) \\
=&\Big(\int_{L_B}\int_{-\infty}^{\infty} \frac{2(Be^{tB}u, e^{tB}u)}{(1+\|e^{tB}u\|^2)^2}\, e^{t(B-I)}u\, dt\, K_B(du), w\Big) \\
=&\int_{L_B}\int_{-\infty}^{\infty} \frac{2(Be^{tB}u, e^{tB}u)}{(1+\|e^{tB}u\|^2)^2}\, (u, e^{t(B^*-I)}w)\, dt\, K_B(du) \\
=&\int_{L_B}\Big(\int_{-\infty}^{\infty} \frac{2(Be^{tB}u, e^{tB}u)}{(1+\|e^{tB}u\|^2)^2}\, dt\Big)(u,w)\, K_B(du) \\
=&c\int_{L_B}(u,w)\, K_B(du),
\end{aligned}$$

where

$$c = \int_{-\infty}^{\infty} \frac{2(Be^{tB}u, e^{tB}u)}{(1+\|e^{tB}u\|^2)^2}\, dt,$$

and substitution

$$s = \|e^{tB}u\|^2$$

gives in view of (18)

$$c = \int_0^{\infty} \frac{ds}{(1+s)^2} = 1.$$

Consequently,

$$\Big(\int_{V_0} \frac{2(Bu,u)}{(1+\|u\|^2)^2} u\, M(du), w\Big) = \int_{L_B} (u,w)\, K_B(du),$$

thus (32) is equivalent to (30) in the continuous case too, and the proof of the theorem has been finished. □

Remark 2. As noted in the Introduction, a condition equivalent to the existence of centering in the continuous case was found in [13]. However, its form there is more complicated and does not fit into our "homogeneous" scheme given in Theorem 13.

The sets Z_A, L_B and the measure K_B depend on the choice of operators A and B. As for the continuous case, it is shown in [2] (cf. also [7, Proposition 4.3.4]) that there is an inner product on V giving rise to a norm $|||\cdot|||$, and a mixing measure K on the unit sphere $L = \{v : |||v||| = 1\}$, such that for every B satisfying (13) we have

$$M(E) = \int_L \int_0^{\infty} \mathbf{1}_E(t^B u)\, \frac{dt}{t^2}\, K(du).$$

In the discrete case, it can also be shown that under a suitable inner product norm we can have the set Z_A independent of $A \in G_a(\mu)$ (though it will still depend on a) (cf. [11]). Denoting this set by Z, we get the formula

$$M(E) = \sum_{n=-\infty}^{\infty} a^{-n} \int_Z \mathbf{1}_E(A^n u)\, M(du)$$

for every A satisfying (2). Theorem 13 may now be given the following form.

Theorem 13'. *The existence of a universal quasi-decomposability centering is equivalent to the conditions:*

(i) *Discrete case*

$$\int_Z (u, w)\, M(du) = 0 \qquad \text{for all } w \in \mathcal{N}(A^* - aI)$$

where A is any *operator satisfying* (2)*;*

(ii) *Continuous case*

$$\int_L (u, w)\, K(du) = 0 \qquad \text{for all } w \in \mathcal{N}(B^* - aI),$$

where B is any *operator satisfying* (13)*.*

Remark 3 (cf. [7, 13])**.** Ordinary multivariate semistable and stable measures are obtained if there is an operator A satisfying (2), or operator B satisfying (13), respectively, being a multiple of the identity. The only problem with the existence of centering in this case arises when $A = aI$ and $B = I$. In such a case we have

$$Z_A = \{v\colon a < \|v\| \leqslant 1\}, \qquad L_B = \{v\colon \|v\| = 1\},$$
$$\mathcal{N}(A^* - aI) = \mathcal{N}(B^* - I) = V,$$

thus the conditions on the existence of centering are respectively:

$$\int_{a<\|u\|\leqslant 1} u\, M(du) = 0, \qquad \int_{\|u\|=1} u\, M(du) = 0.$$

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